Appendix to "Optimal Prizes for All-Pay Contests in Heterogeneous Crowdsourcing"

Tie (Tony) Luo
Institute for Infocomm Research, A*STAR, Singapore 138632

***Abstract*—This addendum provides all the omitted proofs in [Luo et al., 2014].**

A. Proof of Lemma 1

1) Existence: The existence proof parallels [Lebrun, 1999] (Theorem 2) where the assumptions are:

(i) IPV model: the player types (e.g., values of prize) are independent and private;
(ii) Common support: all players share the same interval of types, $[\underline{v}, \bar{v}]$;
(iii) Properties of distribution: all c.d.f. F_i's are continuous over the closed interval $[\underline{v}, \bar{v}]$ and differentiable over the half-open interval $(\underline{v}, \bar{v}]$, and all p.d.f. f_i's are bounded away from zero over $(\underline{v}, \bar{v}]$;[1]
(iv) Mass at the lower extremity: either (a) there is no mass at $\underline{v}$, i.e., $F_i(\underline{v}) = 0, \forall i$, or (b) $F_i(\underline{v}) > 0$ and F_i is right-hand differentiable at $\underline{v}$ and $f_i(v)$ is bounded away from zero for all $v \in [\underline{v}, \bar{v}]$, $\forall i$.

Our model obeys all these assumptions. Moreover, we conjecture that the existence of equilibria in first-price auctions is reciprocal to the existence of equilibria in all-pay auctions, provided that all the assumptions are the same except for the auction institution. (However, monotonicity does not inherit this reciprocity and we need to reconcile a difference in utility functions; see next.)

2) Monotonicity: The monotonicity proof follows [Maskin and Riley, 2000] (Proposition 1), but to apply that result we need to reconcile a difference between first-price and all-pay auctions. In first-price auctions, the payoff of a bidder i is zero when his bid is unsuccessful, but in all-pay auctions, it is negative. Therefore, we will use a "modified" utility function, $\hat{u}_i$, by adding back the negative component (i.e., payment) to the original utility function, u_i, as

$$\hat{u}_i = u_i + p(b_i, v_i).$$

Furthermore, note that the utility referred to by [Maskin and Riley, 2000] is actually the utility when a bidder *wins* the auction, not the *expected* utility that is commonly used and that involves a winning probability. Therefore, we end up using the following modified "winning" utility function:

$$\hat{u}_i^{win} = u_i^{win} + p(b_i, v_i),$$

where u_i^{win} is the original utility when agent i wins the contest, and is thus $V(v_i, Z_i(b_i)) - p(b_i, v_i)$. Hence, $\hat{u}_i^{win} = V(v_i, Z_i(b_i))$. Now, in order to apply the result of [Maskin and Riley, 2000] we need to verify whether $\hat{u}_i^{win}$ satisfies the *weak supermodularity* which is defined as

$$\frac{\partial^2 u_i^{win}(b_i, v_i, v_{-i})}{\partial b_i \partial v_i} \geq 0, \quad \forall i, \forall \boldsymbol{v} = (v_i, v_{-i}).$$

It is reasonable to assume that the value function of a prize, $V(v, Z)$, satisfies $\frac{\partial V(v,Z)}{\partial Z} > 0$ (higher prize implies more value) and $\frac{\partial V(v,Z)}{\partial v} \geq 0$ (higher type is able to derive more value from the same prize). Further, we assume that $\frac{\partial^2 V(v,Z)}{\partial v \partial Z} \geq 0$ which means that, as prize increases, the value that a higher type is able to derive from the prize increases in a faster speed than the prize; in other words, if the prize increases linearly, then a higher type can gain value in a super-linear manner.[2] In addition, since $Z'(b) \geq 0$ which is self-explanatory (higher bids should deserve higher prize), we now have

$$\frac{\partial^2 V(v_i, Z_i(b_i))}{\partial b_i \partial v_i} = \frac{\partial^2 V(v_i, Z_i(b_i))}{\partial Z_i \partial v_i} \frac{\mathrm{d}Z_i}{\mathrm{d}b_i} \geq 0.$$

This means that u_i^{win}, which derives from our utility function, is weakly supermodular. The monotonicity of equilibrium thus follows from [Maskin and Riley, 2000] (Proposition 1).

3) Uniqueness: The uniqueness proof is analogous to [Lebrun, 2006] (Theorem 1) as a special case of possibly different type supports. Four assumptions need to be verified against: the first two are (i) and (iii) in the existence proof above (IPV and distribution function), the third is $F_i(\underline{c}) = 0$, i.e., there is no mass point at the lower extremity (the case with mass point, i.e., $F_i(\underline{c}) > 0$, and common support also admits a unique equilibrium, as proved in [Lebrun, 1999][3]). The fourth and last assumption is that there exists a $\delta > 0$ such that F_i is strictly *log-concave* over $(\underline{v}, \underline{v} + \delta)$.[4]

As our model obeys the first three assumptions, we only need to limit our c.d.f. F_i's to those that satisfy the log-concavity. This means that $\ln F_i$ must be strictly concave,

[1] That is, there exists some $\delta > 0$ such that $f_i(v) > \delta$ for all $v \in (\underline{v}, \bar{v}]$.

[2] This is certainly feasible in practice. For example, a higher prize enables a stronger winner to invest in a wider portfolio with super-linear return, or to attract *much* larger attention from the media. In fact, one could draw an analogy here to the well-known Mathew effect, "the rich get richer and the poor get pooer'.

[3] Uniqueness in the case with a mass point was also proved by [Maskin and Riley, 2003] and [Bajari, 2001]. However, [Lebrun, 2006] points out that both [Bajari, 2001] and [Maskin and Riley, 1996]—an early version of [Maskin and Riley, 2003]—contain an error in their proof.

[4] This last assumption has been tailored to our model by us. In detail, since our model essentially admits a reserve price of zero and adopts a common type support, two of the three "or" conditions (i–iii) postulated by Theorem 1 of [Lebrun, 2006] are violated, and hence we must satisfy the remaining assumption (iii) therein which is the log-concavity stated above.

or f_i/F_i is strictly decreasing. Nevertheless, this additional condition is *not* restrictive, as it is in fact common in economic theory (see [An, 1998] and [Bagnoli and Bergstrom, 2005]), and as an example, both uniform and exponential distributions are log-concave. Also, note that it only requires F_i to be "locally" log-concave, i.e., near the lower extremity $\underline{v}$ and not over the entire support.

4) Common (bid) support: The bids sharing the common support $[0, \bar{b}]$ follows from the combination of Lemma 1 and 4 of [Amann and Leininger, 1996] where the argument of the two lemmas holds for the n-player case. Alternatively, it can also be proved using Lemma 10 and 11 of [Maskin and Riley, 2003] but with a few additional steps for verifying against the assumptions therein.

B. Proof of Proposition 3

The existence and uniqueness are due to Lemma 1.[5] To solve for the equilibrium strategy $\boldsymbol{b} = (b_1, b_2)$, first write agent i's utility below, where we recall that $v_i(\cdot) := \beta_i^{-1}(\cdot)$,

$$u_1 = F_2(v_2(b_1))v_1 - p(b_1),$$
$$u_2 = F_1(v_1(b_2))v_2 - p(b_2).$$

To maximize u_i, applying the first-order condition yields

$$\partial u_1/\partial b_1 = F_2'(v_2(b_1))v_2'(b_1)v_1 - p'(b_1) = 0, \tag{1}$$
$$\partial u_2/\partial b_2 = F_1'(v_1(b_2))v_1'(b_2)v_2 - p'(b_2) = 0. \tag{2}$$

In (3), treat b_2 as a parameter and substitute it by b_1, and meanwhile notice that $v_2 = v_2(b_2)$. Then we have

$$F_1'(v_1(b_1))v_1'(b_1)v_2(b_1) = p'(b_1). \tag{3}$$

Define $k(v_1) := v_2(b_1(v_1)) = \beta_2^{-1}(b_1(v_1))$, in the spirit of [Amann and Leininger, 1996]. Thus

$$k'(v_1) = v_2'(b_1(v_1))b_1'(v_1). \tag{4}$$

The first term on the r.h.s. equals, according to (1),

$$v_2'(b_1(v_1)) = \frac{p'(b_1)}{F_2'(v_2(b_1(v_1)))v_1} = \frac{p'(b_1)}{F_2'(k(v_1))v_1}.$$

The second term can be rewritten firstly using the theorem of derivative of inverse function, and secondly(3), as follows:

$$b_1'(v_1) = \frac{1}{v_1'(b_1(v_1))} = \frac{F_1'(v_1)v_2(b_1)}{p'(b_1)} = \frac{F_1'(v_1)k(v_1)}{p'(b_1)}. \tag{5}$$

Therefore, (4) equals, by replacing v_1 with v,

$$k'(v) = \frac{F_1'(v)k(v)}{F_2'(k(v))v}. \tag{6}$$

Agent 1's equilibrium strategy can now be solved via (5):

$$p'(b_1)b_1'(v_1) = p'_{v_1}(b_1(v_1)) = F_1'(v_1)k(v_1)$$
$$\Rightarrow b_1(v_1) = p^{-1}\Big(\int_{k^{-1}(\underline{v})}^{v_1} F_1'(v)k(v)\,\mathrm{d}v\Big)$$

where $k(v)$ is determined by (6). Using $k^{-1}(\underline{v})$ instead of $\underline{v}$ as the lower limit of integral is to ensure $k(v)$ to be differentiable (cf. (4)) as $k(\cdot)$ essentially maps the support of v_1 to that of v_2. In addition, using $\underline{v}$ in $k^{-1}(\cdot)$ is because the equilibrium strategy is monotone increasing (cf. Lemma 1).

Agent 2's equilibrium strategy is then solved by the definition of $k(\cdot)$, as

$$\beta_2(k(v_1)) = b_1(v_1) \quad \Rightarrow \quad b_2(v_2) = b_1(k^{-1}(v_2)).$$

The boundary condition $k(\bar{v}) = \bar{v}$ can be proved using Lemma 1 as follows. Since the common support of equilibrium bids is $[0, \bar{b}]$ and the strategy is monotone increasing, $b_1(\bar{v}) = \bar{b}$. Furthermore, the inverse function of the strategy is also monotone increasing, and hence $\beta_2^{-1}(\bar{b}) = \bar{v}$. Therefore, it follows from the definition of $k(v)$ that $k(\bar{v}) = \beta_2^{-1}(b_1(\bar{v})) = \bar{v}$.

C. Proof of Proposition 4

The utility of an agent of type v is

$$u = vF^{n-1}(v) - p(b).$$

To maximize u, applying the first-order condition with respect to b, and noting that the inner v is actually $v(b)$, give

$$v\frac{\mathrm{d}F^{n-1}(v)}{\mathrm{d}v}\frac{1}{b'(v)} - p'(b) = 0$$
$$\Rightarrow p'(b)b'(v) = p'_v(b(v)) = v\frac{\mathrm{d}F^{n-1}(v)}{\mathrm{d}v}$$
$$\Rightarrow p(b(v)) = \int_{\underline{v}}^{v} t\,\mathrm{d}F^{n-1}(t) = tF^{n-1}(t)|_{\underline{v}}^{v} - \int_{\underline{v}}^{v} F^{n-1}(t)\,\mathrm{d}t$$
$$\Rightarrow b(v) = p^{-1}\Big(vF^{n-1}(v) - \int_{\underline{v}}^{v} F^{n-1}(t)\,\mathrm{d}t\Big).$$

[5]Alternatively, the existence can be attributed to [Amann and Leininger, 1996] (Theorem 1) and the uniqueness to [Maskin and Riley, 2003] (Proposition 1).

References

[Amann and Leininger, 1996] Amann, E. and Leininger, W. (1996). Asymmetric all-pay auctions with incomplete information: The two-player case. *Games and Economic Behavior*, 14(1):1–18.

[An, 1998] An, M. Y. (1998). Logconcavity versus logconvexity: A complete characterization. *Journal of Economic Theory*, 80(2):350–369.

[Bagnoli and Bergstrom, 2005] Bagnoli, M. and Bergstrom, T. (2005). Log-concave probability and its applications. *Economic Theory*, 26(2):445–469.

[Bajari, 2001] Bajari, P. (2001). Comparing competition and collusion: A numerical approach. *Economic Theory*, 18:187–205.

[Lebrun, 1999] Lebrun, B. (1999). First price auctions in the asymmetric n bidder case. *International Economic Review*, 40(1):125–141.

[Lebrun, 2006] Lebrun, B. (2006). Uniqueness of the equilibrium in first-price auctions. *Games and Economic Behavior*, 55(1):131–151.

[Luo et al., 2014] Luo, T., Kanhere, S. S., Das, S. K., and Tan, H.-P. (2014). Optimal prizes for all-pay contest in heterogeneous crowdsourcing. In *Proc. IEEE MASS*.

[Maskin and Riley, 1996] Maskin, E. and Riley, J. G. (1996). Equilibrium in sealed high bid auctions. Mimeo. Revised version (a) of UCLA Working Paper #407.

[Maskin and Riley, 2000] Maskin, E. S. and Riley, J. G. (2000). Equilibrium in sealed high bid auctions. *Review of Economic Studies*, 67(3):439–454.

[Maskin and Riley, 2003] Maskin, E. S. and Riley, J. G. (2003). Uniqueness of equilibrium in sealed high-bid auctions. *Games and Economic Behavior*, 45(2):395–409.