Online Appendix to: Incentive Mechanism Design for Crowdsourcing: An All-Pay Auction Approach

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A. PROOF OF LEMMA 1

Proof. Proving the existence of equilibrium, particularly for Bayesian games, is a topic that has attracted tremendous effort from mathematicians and economists for decades. Pertaining to our model, we provide three proofs below, from the simplest to the most complex, leveraging three different pieces of prior art.

Proof A: Although the main result of Jackson and Swinkels [2005], Theorem 6, applies to our case, it requires a laborious process of verifying against ten assumptions. Instead, by comparing to the example auction (vi) in Section 3.3 of Jackson and Swinkels [2005], which specifies a single-object all-pay auction with a standard set of basic auction rules, it is straightforward to verify that our model satisfies the rules. Thus, we apply Theorem 2 therein and immediately proves the existence of equilibrium. To prove the decreasing monotonicity, we note that the agent type in our case can be viewed as the inverse of prize valuation. Then the result immediately follows from the increasing monotonicity of a standard Bayesian game with agent type being valuation (which can be found in any auction textbook, such as Krishna [2009]).

Proof B: We prove the result by finding a sufficient condition for the existence of a symmetric and decreasing equilibrium, following the spirit of Krishna and Morgan [1997] but dropping the assumption about type affiliation. To begin with, define $x_i := 1/s_i$ to be each agent's "inverse type," and denote the corresponding agent belief by $F_X(x)$ as compared to $F(s)$, and bidding strategy by $b(x)$ as compared to $z(s)$. This leads to an equivalent model to ours, and we proceed to prove the existence and *increasing* monotonicity for this new model instead.

Consider an arbitrary agent i and define $Y_1 := \max_{j\neq i}\{x_j\}$, which is a random variable unknown to i. Define $\Psi : \mathbb{R}^2 \to \mathbb{R}$ as $\Psi(x, y) := v(x, y) f_{Y_1}(y|x)$, where $v(x, y) := \mathbb{E}[M(b)|x_i = x, Y_1 = y]$ is the expected prize value to i given that his (inverse) type is x and $Y_1 = y$, and $f_{Y_1}(y|x)$ is the density of Y_1 given $x_i = x$. This is a rather general definition taking into account possible affiliation between agent types. As our case conforms to an IPV model, the function Ψ can be simplified to $\Psi(x, y) = \mathbb{E}[M(b)|x_i = x] f_{Y_1}(y)$. As bid $b(\cdot)$ is increasing in type x [Krishna 2009], $M(b)$ is increases in x because a higher bid should lead to a higher prize. Therefore, $\Psi(x, y)$ is increasing in x, and it then follows from Theorem 2 of Krishna and Morgan [1997] that a symmetric equilibrium exists, which is the (implicit) solution to the equation

$$b(x) = \int_{-\infty}^{x} \mathbb{E}[M(b)|x_i = t]\, f_{Y_1}(t)\, \mathrm{d}t, \tag{A.1}$$

where $x = 1/s$.

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Proof C: This proof is based on Theorem 7 in Athey [2001], which is further based on Theorem 6 therein that states the following: a pure strategy Nash equilibrium exists in increasing (in our case, decreasing) strategies if the auction game satisfies the Spence-Mirrlees single-crossing property (SCC) and assumptions A1–A3 in Athey [2001].

As our all-pay auction is under an IPV setting and bidders are (weakly) risk averse, it satisfies SCC and A3 according to Theorem 7. Furthermore, as the strategy space of each agent is continuous and bounded (for otherwise it will lead to negative payoff), we are left with A1 and A2 to verify against.

In our model, the agent types have a joint density with respect to Lebesgue measure, $f^{\tilde{n}}(s)$, that is bounded and atomless, where $f(s) = F'(s)$. Furthermore, because $\tilde{\pi}_i$ and $f(s_{-i})$ are bounded, the expected payoff $\pi_i = \int_{[\underline{s},\bar{s}]^{\tilde{n}-1}} \tilde{\pi}_i(s_i, z_i^*, \mathbf{z}_{-i}^*(s_{-i})) f(s_{-i})\,\mathrm{d}s_{-i}$ is finite over the convex support $[\underline{s}, \bar{s}]^{\tilde{n}-1}$ for all monotone (decreasing) functions $z_j^*(s_j), j \neq i$. Thus, A1 is satisfied.

It is obvious that both the winner's and the losers' payoffs are bounded and continuous. The continuum of strategy space also allows all the agents to have other options besides their respective equilibrium strategies. In addition, bidding higher reduces the gain from winning, while the gain is decreasing in agent type (which is marginal cost in our case). Last, the slope (with respect to type) of the gain to winning, denoted by ρ which is negative, is bounded away from zero, that is, $\exists\lambda > 0$ such that $\rho \leq -\lambda$. Thus, A2 is satisfied. □

B. PROOF OF LEMMA 2

Proof. Because each agent is playing her best response in equilibrium, her equilibrium payoff π^* is also the solution to the optimization problem $\max_z \pi(s, \mathbf{z})$. Furthermore, π^* is also an outcome of all agents playing equilibrium strategies, that is, $\pi^* = \pi(s, \mathbf{z}^*(s))$. Therefore, we apply the envelope theorem [Milgrom and Segal 2002] to (7) with respect to parameter s:[1]

$$\frac{\partial \pi^*}{\partial s} = -P(s)\big[u'(\alpha^*)h(z^*) - u'(-\beta^*)h(z^*)\big] - u'(-\beta^*)h(z^*). \tag{A.2}$$

Integrating the l.h.s. of (A.2) from s to $\bar{s}$ leads to

$$\int_s^{\bar{s}} \frac{\partial \pi^*}{\partial s}\,\mathrm{d}s = \pi^*(\bar{s}) - \pi^*(s) = -\pi^*(s).$$

Here, $\pi^*(\bar{s}) = 0$ as is commonly assumed in the literature, based on the rationale that the principal would like the highest-cost agent to reap zero surplus so that he is indifferent to participating.

Also integrating the right-hand side of (A.2) from s to $\bar{s}$, equals the negative left-hand side of (8). Therefore, by noticing that $\pi^* = \pi(s, \mathbf{z}^*)$, (8) is proved by combining with (7). □

[1]When applying the envelope theorem [Milgrom and Segal 2002] with respect to s, the strategy z^* is treated as fixed instead of a function of s, and the s in $F(s)$ as inside $P(s)$ (6) is treated as an inverse function of z^* and is hence also fixed; only the s incurring the true cost (as in $h(z^*)s$) is considered the true type and differentiable.

C. PROOF OF LEMMA 3

PROOF. Given a symmetric equilibrium of agent strategy profile $\mathbf{z}^*$, the profit of the principal (3) can be rewritten as

$$\Omega(\tilde{n}, \mathbf{z}^*) = \tilde{n} \int_{\underline{s}}^{\bar{s}} z^*(s)\, \mathrm{d}F(s) - M\big(z^{*(\tilde{n})}_{(1)}\big). \tag{A.3}$$

In equilibrium, the winner must be the agent of the lowest type (e.g., marginal cost), that is, $z^{*(\tilde{n})}_{(1)} = z^*(s^{(\tilde{n})}_{(\tilde{n})})$, where $s^{(\tilde{n})}_{(\tilde{n})}$ is the $\tilde{n}$-th order (minimum) of all the types. Its distribution can be determined as

$$\Pr\left(s^{(\tilde{n})}_{(\tilde{n})} \le s\right) = 1 - \Pr\left(s^{(\tilde{n})}_{(\tilde{n})} > s\right) = 1 - \prod_i \Pr(s_i > s) = 1 - (1 - F(s))^{\tilde{n}},$$

where the last two steps are based on that agent types are independent and identically distributed. Therefore,

$$M\big(z^{*(\tilde{n})}_{(1)}\big) = \int_{\underline{s}}^{\bar{s}} M(z^*(s))\, \mathrm{d}\big[1 - (1 - F(s))^{\tilde{n}}\big]. \tag{A.4}$$

Thus, we rewrite the expected profit (4) using (A.3) and (A.4) as

$$\begin{aligned}\Omega^* &= \sum_n np_n \int_{\underline{s}}^{\bar{s}} z^*\, \mathrm{d}F - \sum_n p_n \int_{\underline{s}}^{\bar{s}} M(z^*(s))\, \mathrm{d}\big[1 - (1 - F)^n\big] \\ &= \sum_n np_n \int_{\underline{s}}^{\bar{s}} \big[z^* - M(z^*)(1 - F)^{n-1}\big]\, \mathrm{d}F. \quad \square\end{aligned}$$

D. PROOF OF THEOREM 1

The α_{rn} and β_{rn} in (15) contain $\mathring{M}_{rn}(\cdot)$ and $\mathring{z}_{rn}$, which are spelled out in (17) and (18), respectively.

PROOF. Substituting (10) into (8) gives

$$\begin{aligned}&\int_{s}^{\bar{s}} \big[\epsilon P(s_1)\big(u_1'(\alpha^*_{rn}) - u_1'(-\beta^*_{rn})\big) + 1 + \epsilon u_1'(-\beta^*_{rn})\big] \times h(z^*)\, \mathrm{d}s_1 \\ &\quad = P(s)M(z^*) + \epsilon P(s)\big[u_1(\alpha^*_{rn}) - u_1(-\beta^*_{rn})\big] - \beta^* + \epsilon u_1(-\beta^*_{rn}), \\ &\quad \Rightarrow \int_{s}^{\bar{s}} B^*(s_1) h(z^*)\, \mathrm{d}s_1 = P(s)M(z^*) + A^*(s) - \beta^* \end{aligned} \tag{A.5}$$

where

$$\begin{aligned}A^*(s) &= \epsilon P(s)\big[u_1(\alpha^*_{rn}) - u_1(-\beta^*_{rn})\big] + \epsilon u_1(-\beta^*_{rn}), \\ B^*(s) &= \epsilon P(s)\big[u_1'(\alpha^*_{rn}) - u_1'(-\beta^*_{rn})\big] + \epsilon u_1'(-\beta^*_{rn}) + 1\end{aligned} \tag{A.6}$$

are functions of s only, given z^*_{rn} and $M_{rn}(\cdot)$. Hence,

$$M(z^*(s)) = \frac{1}{P(s)} \left(\beta^* - A^*(s) + \int_{s}^{\bar{s}} B^*(s_1) h(z^*(s_1))\, \mathrm{d}s_1\right). \tag{A.7}$$

Substituting (A.7) into (9) yields

$$\Omega^* = \int_{\underline{s}}^{\bar{s}} \left[z^* \sum_n np_n - \frac{\sum_n np_n(1-F)^{n-1}}{\sum_n p_n(1-F)^{n-1}} \times \left(\beta^* - A^*(s) + \int_s^{\bar{s}} B^*(s_1)h(z^*)\,\mathrm{d}s_1 \right) \right] \mathrm{d}F$$
$$= \int_{\underline{s}}^{\bar{s}} \left[az^*F' - G'(s)\,(\beta^* - A^*(s)) \right] \mathrm{d}s - \int_{\underline{s}}^{\bar{s}} \left[G'(s) \int_s^{\bar{s}} B^*(s_1)h(z^*)\,\mathrm{d}s_1 \right] \mathrm{d}s,$$

where we note that z^* is no longer a function of n as in the usual deterministic population setting. Integrating the second term by parts leads to

$$\left(G(s) \int_s^{\bar{s}} B^*(s_1)h(z^*)\,\mathrm{d}s_1 \right) \Big|_{\underline{s}}^{\bar{s}} - \int_{\underline{s}}^{\bar{s}} G(s)(-B^*(s)h(z^*))\,\mathrm{d}s = \int_{\underline{s}}^{\bar{s}} G(s)B^*(s)h(z^*)\,\mathrm{d}s.$$

Thus,

$$\Omega^* = \int_{\underline{s}}^{\bar{s}} \left[az^*F' - G'(s)(\beta^* - A^*(s)) - G(s)B^*(s)h(z^*) \right] \mathrm{d}s. \tag{A.8}$$

Now, revisiting the principal's objective, $\max_{M(\cdot)} \Omega^*$ (4), we note that choosing an optimal prize function $M(\cdot)$ to (indirectly) maximize profit is essentially inducing an optimal equilibrium strategy to (directly) maximize profit. Therefore, solving $\max_{M(\cdot)} \Omega^*$ is equivalent to solving $\max_{z^*} \Omega^*$. Thus, denoting the integrand in (A.8) by γ, we write its first-order condition (FOC), with respect to z^* and fixing s, as

$$\frac{\partial \gamma}{\partial z^*} = aF' - sG'(s)\frac{\mathrm{d}h(z^*)}{\mathrm{d}z^*} - G(s)B^*(s)\frac{\mathrm{d}h(z^*)}{\mathrm{d}z^*} = 0. \tag{A.9}$$

From the above, we obtain (13) which is the (optimal) equilibrium strategy (and hence we write $\mathring{z}$ in place of z^*, and $\mathring{B}$ in place of B^*).

Next in (A.7), replacing s by $\mathring{s}(z)$, A^* by $\mathring{A}$, and converting the limits of integral from s to z, lead to the optimal prize function (12).

Then, replacing z^*, β^*, A^*, B^* in (A.8) with their corresponding optimal equilibrium variables obtains the maximum profit (14).

In addition, for the solution to FOC (A.9) to exist and be unique, we require $\frac{\partial^2 \gamma}{\partial z^{*2}} < 0$, which implies $[G'(s)s + G(s)B^*(s)]h''(z^*) > 0$. Under weak risk aversion, $\epsilon \ll 1$, so $B^*(s) > 0$. Furthermore, because $G'(s) > 0$ and $G(s) > 0$, it requires $h'' > 0$ (a convex $h(\cdot)$ implies that contributing at a higher level is more costly than at a lower level, which is often the case in reality). □

E. PROOF OF THEOREM 2

Proof. Under weak risk aversion which subsumes risk neutrality, (8) can be rewritten as (A.5) (cf. proof of Theorem 1). Furthermore, we know that (8) is actually $\pi^*(s)$ (cf. proof of Lemma 2). Thus,

$$\pi^*(s) = \int_s^{\bar{s}} B^*(s_1)h(z^*(s_1))\,\mathrm{d}s_1.$$

Rearrange $B^*(s)$, as defined in (A.6), as

$$B^*(s) = \epsilon \left[P(s)u_1'(\alpha_{rn}^*) + (1 - P(s))u_1'(-\beta_{rn}^*) \right] + 1.$$

For risk-neutral agents, $B^{*}(s) = 1$ because $\epsilon = 0$. For weakly risk-averse agents, $u_1' > -\frac{1}{\epsilon}$, and hence

$$B^{*}(s) > \epsilon\left[-\frac{P(s)}{\epsilon} - \frac{1-P(s)}{\epsilon}\right] + 1 = 0.$$

Since $h(z) > 0$ if and only if $z > 0$, it holds for such agents that $\pi^{*} > 0$ if only only if $z^{*} > 0$. That is, any such agent who contributes (any nonzero amount) will reap in equilibrium strictly positive expected payoff. □

F. A TECHNICAL LEMMA

This lemma is used by Appendix G.

LEMMA A (RN: EQUILIBRIUM STRATEGY). *In an all-pay auction with incomplete information and a stochastic population of* **risk-neutral** *agents, given a contribution-dependent prize function $M_{rn}(z)$, the equilibrium strategy $z^{*}_{rn}(s)$ is determined by*

$$\int_{s}^{\bar{s}} h(z^{*}_{rn}(t))\,\mathrm{d}t = M_{rn}(z^{*}_{rn})P(s) - h(z^{*}_{rn})s. \tag{A.10}$$

PROOF. Because $z^{*}_{rn} = \arg\max_z \pi_{rn}(s, z)$, it follows from (7) that

$$\pi^{*}_{rn} = M_{rn}(z^{*}_{rn})P(s) - h(z^{*}_{rn})s, \tag{A.11}$$

where we have let $u(x) = x$ due to risk neutrality. Further, because $\pi^{*}_{rn} = \max_z \pi_{rn}(z)$, we apply the envelope theorem to (A.11) and obtain $\frac{\partial \pi^{*}_{rn}}{\partial s} = -h(z^{*}_{rn})$. Integrating both sides and using the boundary condition ($\pi^{*}_{rn}(\bar{s}) = 0$) lead to $-\pi^{*}_{rn}(s) = -\int_{s}^{\bar{s}} h(z^{*}_{rn}(s_1))\,\mathrm{d}s_1$. Combining this with (A.11) proves (A.10).

Alternatively, we can prove the result using Lemma 2 as a shortcut. When agents are risk neutral, we have $u(x) = x$, $u'(x) = 1$. Substituting this into (8) (in Lemma 2) gives (A.10). □

G. PROOF OF PROPOSITION 2

PROOF. Using Lemma A from Appendix F, substitute M_0 for $M_{rn}(\cdot)$ in (A.10) and then differentiate (A.10) with respect to s, yielding

$$\begin{aligned} -h(z^{*}_{rn,con}(s)) &= M_0 P'(s) - h(z^{*}_{rn,con}) - s\frac{\mathrm{d}h(z^{*}_{rn,con})}{\mathrm{d}s} \\ \therefore \quad \frac{\mathrm{d}h(z^{*}_{rn,con})}{\mathrm{d}s} &= M_0\frac{P'(s)}{s}. \end{aligned}$$

Integrating both sides from s to $\bar{s}$ gives

$$h(z^{*}_{rn,con}(\bar{s})) - h(z^{*}_{rn,con}) = M_0\int_{s}^{\bar{s}} \frac{P'(s_1)}{s_1}\,\mathrm{d}s_1,$$

which leads to (26) because $h(z^{*}_{rn,con}(\bar{s})) = h(0) = 0$.[2] □

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[2]With constant prize, the equilibrium bid $z^{*}_{*,con}(\bar{s})$ is zero, but this does not necessarily hold in the adaptive prize case, where the prize could go to infinity.

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